* UNIT - II *

* Averages of Arithmetic functions *

Definition:- Big-oh - Notation

If $f(x), g(x) > 0$ be two functions $\forall$ $x \geq a$ then we write $f(x) = O[g(x)]$ if the quotient function $\frac{f(x)}{g(x)}$ is bounded for all $x \geq a$ i.e., there exist a constant $M > 0$ such that $\left|\frac{f(x)}{g(x)}\right| \leq M \Rightarrow$

$|f(x)| \leq M\,|g(x)|$

$\rightarrow$ An equation of the form $f(x) = h(x) + O\,g(x)$

$\Rightarrow f(x) - h(x) = O\,g(x)$

$\rightarrow f(x) = O[g(x)] \Rightarrow \int_x^a f(x)\,dx = O\left[\int_x^a g(x)\,dx\right]$

Formulas:-

1. $O(\text{zero}) = \text{zero}$
2. $O[g(x)] = O[g(x)]$
3. $n \cdot O[g(x)] = O[g(x)]$
4. $2 \cdot O[g(x)] = O[g(x)]$
5. $O[g(x)] + O[g(x)] = O[g(x)]$
6. $a + O(a) = O(a)$
7. $2a + O(a) = O(a)$
8. $O[O(a)] = O(a)$
9. $na + O(a) = O(a)$
10. $[y] = y + O(1)$
11. $[\sqrt{x}] = O[\sqrt{x}]$
12. $O\left[\sum_{d \leq x} 1\right] = O(x)$
13. $ca = O(a)$
14. $ca + O(\alpha) = O(\alpha)$
15.

16

17

18 The division function, $d(n) = \sum_{d|n} 1$

19 $O(a) = a^n$

20. $O(1) + O(a) = O(a)$

21 $\sum_{d \le x} O\left(\frac{x}{d}\right) = O\left[x \cdot \sum_{d \le x} \frac{1}{d}\right] = O\left[\sum_{d \le x} \frac{x}{d}\right]$

22. $|\mu(1)| = 1$

23 $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

24 $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$

25. $O(1) + 1 = O(1)$

26 $O(1) \cdot O(1) = O(1)$

27. $\sum_{d \le x} \mu(d) = 1$

28 If $f_1(x) = O(g_1(x))$ and $f_2 = O(g_2(x))$ then

$f_1(x) + f_2(x) = O(g_1(x)) + O(g_2(x))$

$= O[g_1(x) + g_2(x)]$

$f_1(x) \cdot f_2(x) = O(g_1(x)) \cdot O(g_2(x))$

$= O[g_1(x) \cdot g_2(x)]$

=

Euler Summation Formula

Statement:- If 'f' has a continuous derivative f' on the interval $[y, x]$ where $0 < y < x$ then

$$\sum_{y < n \le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t - [t]) f'(t)\,dt + ([x] - x) f(x) - ([y] - y) f(y)$$

Proof :- Let $[y] = m$, $[x] = k$

Divide the interval $[y, x]$ into sub-intervals

Consider the sub intervals $(n-1, n)$ and take $[t]_{n-1}$

consider $\int_{n-1}^{n} [t] f'(t)\,dt = \int_{n-1}^{n} (n-1) f'(t)\,dt$

$$= (n-1)\int_{n-1}^{n} f'(t)\,dt$$

$$= (n-1)\left[f(n) - f(n-1)\right]$$

$$= (n-1)f(n) - (n-1)f(n-1)$$

$$= nf(n) - f(n) - (n-1)f(n-1)$$

$$= \left[nf(n) - (n-1)f(n-1)\right] - f(n)$$

Summing from $n = m+1$ to k on the both sides

$$\sum_{n=m+1 \text{ to } k}^{k} \int_{n-1}^{n} [t] f'(t)\,dt = \sum_{n=m+1}^{k} \left[\left[nf(n) - (n-1)f(n-1)\right] - f(n)\right]$$

$$= \sum_{n=m+1}^{k} nf(n) - \sum_{n=m+1}^{k} (n-1)f(n-1) - \sum_{n=m+1}^{k} f(n)$$

$$\Rightarrow \int_{m}^{m+1} [t] f'(t)\,dt + \int_{m+1}^{m+2} [t] f'(t)\,dt + \cdots$$

$$+ \int_{k-1}^{k} [t] f'(t)\,dt = \left[(m+1)f(m+1) + (m+2)f(m+2) + \cdots + kf(k)\right] -$$

$$mf(m) - (m+1)f(m+1) - \cdots - (k-1)f(k-1) - \sum_{n=m+1}^{k} f(n)$$

$$\int_{m}^{k} [t] f'(t)\,dt = kf(k) - mf(m) - \sum_{n=m+1}^{k} f(n)$$

$m+1 < n \le k$

$\Rightarrow m < m+1 < n \le k$

$[y] = m < n \le k$

$[y] \le [x]$ $y < n \le [x]$

$$\int_{y}^{x} [t] f'(t)\,dt = [x]f(x) - [y]f(y) - \sum_{y<n\le x} f(n)$$

$$\sum_{y<n\le x} f(n) = [x]f(x) - [y]f(y) - \int_{y}^{x} [t] f'(t)\,dt \quad \text{--- (1)}$$

consider $\int_y^x 1\cdot f'(t)\,dt = \Big[t\,[f(t)]\Big]_y^x - \int_y^x t\,f'(t)\,dt$

$$= x f(x) - y f(y) - \int_y^x t f'(t)\,dt$$

$$\int_y^x f(t)\,dt - x f(x) + y f(y) + \int_y^x t\, f'(t)\,dt = 0 \quad \text{--- (2)}$$

from adding (1) and (2) we have

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x \{t - [t]\} f'(t)\,dt + \{[x]-x\} f(x) - \{[y]-y\} f(y)$$

Sec: Some Elementary Asymptotic functions

the euler constant $C = \lim_{n\to\infty}\left[\left[1+\frac{1}{2}+\cdots\frac{1}{n}\right]-\log n\right]$

(or)

$$= \lim_{n\to\infty}\left[\sum_{n\leq x}\frac{1}{n}-\log n\right]$$

Riemann zeta function:-

$$\xi(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \quad \text{if } s>1$$

$$\xi(s) = \lim_{x\to\infty}\left[\sum_{n\leq x}\frac{1}{n^s} - \frac{x^{1-s}}{1-s}\right] \quad \text{if } 0<s<1$$

Theorem:-

(i) If $x\geq 1$, $\sum_{n\leq x}\frac{1}{n} = \log x + C + O\left[\frac{1}{x}\right]$

(ii) $\sum_{n\leq x}\frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \xi(s) + O(x^{-s})$ for $s>0, s<1$

(iii) $\sum_{n\geq x}\frac{1}{n^s} = O(x^{1-s})$ for $s>1$

(iv) $\sum_{n\leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$ if $\alpha\geq 0$

Proof:-

(i) By euler summation formula, we have

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x [t-[t]] f'(t)\,dt + \{[x]-x\} f(x) -$$

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$([y]-y)f(y)$.

Take $y=1$ and $f(n)=\frac{1}{n}$

$$f'(n)=\frac{-1}{n^2}$$

$$\therefore \sum_{y<n\le x}\frac{1}{n}=\int_1^x \frac{1}{t}dt+\int_1^x (t-[t])\left(\frac{-1}{t^2}\right)dt+\frac{([x]-x)}{x}+o(1)$$

$$=[\log t]_1^x+\int_1^\infty (t-[t])\left(\frac{-1}{t^2}\right)dt+\int_\infty^x (t-[t])\left(\frac{-1}{t^2}\right)dt$$

$$+\frac{[x]-x}{x}+o(1)$$

$$=\log x+\int_1^\infty (t-[t])\left(\frac{-1}{t^2}\right)dt+\int_x^\infty (t-[t])\left(\frac{1}{t^2}\right)dt+$$

$$\frac{[x]-x}{x}+o(1)$$

consider $\left|\frac{[x]-x}{x}\right|\le\frac{1}{x}$

since $\frac{[x]-x}{x}=o\left[\frac{1}{x}\right]$

consider $\left|\int_x^\infty \frac{t-[t]}{t^2}\right|dt\le\left|\int_x^\infty \frac{1}{t^2}dt\right|$

$$\le\left|\left(\frac{-1}{t}\right)_x^\infty\right|$$

$$\le\frac{1}{x}$$

$$\therefore \int_x^\infty\left(\frac{t-[t]}{t^2}\right)dt=o\left(\frac{1}{x}\right)$$

$$\sum_{n\le x}\frac{1}{n}=\log x+\int_1^\infty (t-[t])\left(\frac{-1}{t^2}\right)dt+o\left(\frac{1}{x}\right) \quad \text{—— (1)}$$

From Euler's constant, we have

$$c=\lim_{n\to\infty}\left[1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right] \text{ (or)}$$

$$=\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n}-\log x\right)$$

$$=\lim_{x\to\infty}\left[\cancel{\log x}+\int_1^\infty (t-[t])\left(\frac{-1}{t^2}\right)dt+o\left(\frac{1}{x}\right)-\cancel{\log x}\right]$$

$$=\lim_{x\to\infty}\int_1^\infty (t-[t])\left(\frac{-1}{t^2}\right)dt+o\left(\frac{1}{x}\right)$$

let $c=\int_1^\infty (t-[t])\left(\frac{-1}{t^2}\right)dt$

substitute 'c' value in ①

$$\therefore \sum_{n\le x} \frac{1}{n} = \log x + c + o\left(\frac{1}{x}\right)$$

(ii) By Euler summation formula, we have

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-[t]) f'(t)\,dt + ([x]-x) f(x) - ([y]-y) f(y).$$

Take $y=1$ and $f(n) = \frac{1}{n^s} \Rightarrow f'(n) = \frac{-s}{n^{s+1}}$

$$\therefore \sum_{n\le x} \frac{1}{n^s} = \int_1^x \frac{1}{t^s}\,dt + \int_1^x (t-[t])\left(-\frac{s}{t^{s+1}}\right) dt + \frac{[x]-x}{x^s} +$$

$$= \left[\frac{t^{1-s}}{1-s}\right]_1^x + \int_1^\infty (t-[t])\left(\frac{-s}{t^{s+1}}\right) dt + \int_\infty^x (t-[t]) \left(\frac{-s}{t^{s+1}}\right) dt + \frac{[x]-x}{x^s} + o(1)$$

$$= \frac{x^{1-s}}{1-s} - \frac{1}{1-s} + \int_1^\infty (t-[t])\left(\frac{-s}{t^{s+1}}\right) dt + \int_x^\infty (t-[t]) \left(\frac{s}{t^{s+1}}\right) dt + o(1) + \frac{[x]-x}{x^s}$$

consider $\left|\frac{[x]-x}{x^s}\right| \le \left|\frac{1}{x^s}\right|$

$$\therefore \frac{[x]-x}{x^s} = o\left(\frac{1}{x^s}\right)$$

consider $\left|\int_x^\infty \frac{s(t-[t])}{t^{s+1}}\,dt\right| \le \int_x^\infty \left|\frac{s}{t^{s+1}}\right| dt$

$$\le \left|\frac{s t^{-s-1+1}}{-s-1+1}\right|_x^\infty$$

$$\le \left|\frac{\cancel{s} t^{-s}}{-\cancel{s}}\right|_x^\infty$$

$$\le \left|\frac{-1}{t^s}\right|_x^\infty$$

$$\le \left|\frac{1}{t^s}\right|_x^\infty$$

$$= o\left(\frac{1}{x^s}\right)$$

$$\therefore \int_x^\infty \frac{s(t-[t])}{t^{s+1}}\,dt = o\left(\frac{1}{x^s}\right)$$

$$\therefore \sum_{n\le x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}-\frac{1}{1-s}+\int_1^\infty (t-[t])\left(\frac{-s}{t^{s+1}}\right)dt+o\left(\frac{1}{x^s}\right)+o(1) \longrightarrow (1)$$

By Rie-mann zeta-function, we have

$$\zeta(s)=\lim_{x\to\infty}\left[\sum_{n\le x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}\right]$$

$$=\lim_{x\to\infty}\left[\left[\frac{x^{1-s}}{1-s}-\frac{1}{1-s}\right]+\int_1^\infty (t-[t])\left(\frac{-s}{t^{s+1}}\right)dt+o\left(\frac{1}{x^s}\right)+o(1)-\frac{x^{1-s}}{1-s}\right]$$

$$=\lim_{x\to\infty}\frac{-1}{1-s}+\int_1^\infty (t-[t])\left[\frac{-s}{t^{s+1}}\right]dt+o(1)$$

$$\therefore \zeta(s)=\frac{-1}{1-s}+\int_1^\infty (t-[t])\frac{-s}{t^{s+1}}dt+o(1)$$

substitute $\zeta(s)$ value in equation (1)

$$\therefore \sum_{n\le x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}+\zeta(s)+o(x^{-s})$$

(iii) By euler's summation formula, we have

$$\sum_{y<n\le x}f(n)=\int_y^x f(t)dt+\int_y^x (t-[t])f'(t)dt+([x]-x)f(x)-([y]-y)f(y)$$

we know that $\zeta(s)=\sum_{n=1}^\infty \frac{1}{n^s}$ if $s>1$

$$=\sum_{n\le x}\frac{1}{n^s}+\sum_{n>x}\frac{1}{n^s}$$

$$\sum_{n>x}\frac{1}{n^s}=\zeta(s)-\sum_{n\le x}\frac{1}{n^s}$$

$$=\zeta(s)-\left[\frac{x^{1-s}}{1-s}+\zeta(s)\right]+\left[o(x^{-s})\right]$$

$$=\frac{-x^{1-s}}{1-s}+o(x^{-s})$$

consider $\left|\frac{x^{1-s}}{1-s}\right|<|x^{1-s}|$

$$\therefore \left|\frac{-x^{1-s}}{s-1}\right|=o(x^{1-s})$$

$$\therefore \sum_{n>x} \frac{1}{n^s} = O(x^{1-s}) + O(x^{-s})$$

$$\therefore \sum_{n>x} \frac{1}{n^s} = O(x^{1-s})$$

(iv) By euler's summation formula, we have

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-[t]) f'(t)\,dt + [([x]-x) f(x) - ([y]-y) f(y)]$$

Take $y=1$ and $f(n)=n^\alpha$

$$\Rightarrow f'(n) = n^{\alpha-1}\alpha$$

$$\sum_{n\le x} n^\alpha = \int_1^x t^\alpha\,dt + \int_1^x (t-[t])\alpha t^{\alpha-1}\,dt - ([x]-x)x^\alpha + O(1)$$

$$= \left[\frac{t^{\alpha+1}}{\alpha+1}\right]_1^x + \int_1^\infty (t-[t])\alpha t^{\alpha-1}\,dt + \int_x^\infty (t-[t])[-\alpha t^{\alpha-1}]\,dt + [[x]-x]x^\alpha + O(1)$$

$$= \frac{x^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} + \int_1^\infty (t-[t])\alpha t^{\alpha-1}\,dt + \int_x^\infty (t-[t])(-\alpha t^{\alpha-1})\,dt + ([x]-x)x^\alpha + O(1)$$

consider $|([x]-x)x^\alpha| \le |x^\alpha|$

$$\therefore ([x]-x)x^\alpha = O(x^\alpha)$$

$$= \frac{x^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} + \int_1^\infty (t-[t])\alpha t^{\alpha-1}\,dt + \int_x^\infty (t-[t])(-\alpha t^{\alpha-1})\,dt + O(x^\alpha) + O(1)$$

consider $\left|\int_x^\infty (t-[t])(-\alpha t^{\alpha-1})\,dt\right| \le \int_x^\infty |(-\alpha t^{\alpha-1})|\,dt$

$$\le \int_x^\infty |-\alpha t^{\alpha-1}|\,dt$$

$$\le \left|-\alpha \frac{t^{\alpha-1+1}}{\alpha-1+1}\right|_x^\infty$$

$$\le |-t^\alpha|_x^\infty$$

$$\therefore \int_x^\infty |(t-[t])(-\alpha t^{\alpha-1})\,dt| \le O(x^\alpha)$$

$$\therefore \int_x^\infty (t-[t])(-\alpha t^{\alpha-1}) = O(x^\alpha)$$

$$= \frac{x^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} + 1\int_1^\infty (t-[t])(\alpha t^{\alpha-1})\,dt + O(x^\alpha) + O(x^\alpha) + O(1)$$

$$= \frac{x^{\alpha+1}}{\alpha+1} + O(1) + O(1) + O(x^\alpha) + O(1)$$

$$\therefore \sum_{n\le x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$$

1 Prove that $\sum_{n\le x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + A\,(\text{constant}) + O\left[\frac{\log x}{x}\right]$

Proof:- choose $f(n) = \frac{\log n}{n}$

$f(t) = \frac{\log t}{t}$, $f'(t) = \frac{t\cdot\frac{1}{t} - \log t}{t^2} = \frac{1-\log t}{t^2}$

By euler's summation formula, we have

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-[t])f'(t)\,dt + ([x]-x)f(x) - ([y]-y)f(y)$$

$$\therefore \sum_{1<n\le x} \frac{\log n}{n} = \int_1^x \frac{\log t}{t}\,dt + \int_1^\infty (t-[t])\left[\frac{1-\log t}{t^2}\right]dt + \int_\infty^x (t-[t])\left[\frac{1-\log t}{t^2}\right]dt + \frac{[x]-x}{x}\log x - ([y]-y)\frac{\log y}{y}$$

$$= \int_1^x \frac{\log t}{t}\,dt + \int_1^\infty (t-[t])\frac{1-\log t}{t^2}\,dt - \frac{(x-[x])}{x}\log x - \int_x^\infty (t-[t])\frac{1-\log t}{t^2}\,dt + O(1)$$

$$= \frac{1}{2}\log^2 x + \int_1^\infty (t-[t])\frac{1-\log t}{t^2}\,dt - \int_x^\infty (t-[t])\frac{1-\log t}{t^2}\,dt - \frac{(x-[x])}{x}\log x + O(1)$$

$$\therefore \left|\int_x^\infty (t-[t])\frac{1-\log t}{t^2}\right| \le \int_x^\infty \frac{1-\log t}{t^2}\,dt$$

$$\le \int_x^\infty d\left(\frac{\log t}{t}\right)$$

$$\leq \left[\frac{\log t}{t}\right]_{x}^{\infty}$$

$$= \frac{\log x}{x} = O\left(\frac{\log x}{x}\right)$$

$$\therefore \sum_{y<n\leq x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + \int_1^{\infty} (t-[t])\left[\frac{1-\log t}{t^2}\right] dt + O\left(\frac{\log x}{x}\right) + O(1)$$

$$\sum_{y<n\leq x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + A(\text{constant}) + O\left(\frac{\log x}{x}\right)$$

2. Prove that $\sum_{2\leq n\leq x} \frac{1}{n\log n} = \log(\log x) + B(\text{constant}) + O\left[\frac{1}{x\log x}\right]$

Proof:- choose $f(n) = \frac{1}{n\log n}$; $y = 2$

$f(t) = \frac{1}{t\log t}$; $f'(t) = \left[\frac{1}{t\log t}\right]'$

By Euler's summation formula, we have

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-[t]) f'(t)\,dt + ([x]-x)f(x) - ([y]-y)f(y)$$

$$\sum_{2<n\leq x} \frac{1}{n\log n} = \int_2^x \frac{1}{t\log t} dt + \int_2^{\infty} (t-[t])\left[\frac{1}{n\log n}\right]' dt + \int_{\infty}^{x} (t-[t])\left[\frac{1}{n\log n}\right]' dt + \left(\frac{[x]-x}{x}\right)\frac{1}{\log x} - \left[\frac{[y]-y}{y}\right]\log y.$$

$$= \int_2^x \frac{1}{t\log t} dt + \int_2^{\infty} (t-[t])\left(\frac{1}{n\log n}\right)' dt + \int_{\infty}^{x} (t-[t])\left[\frac{1}{n\log n}\right]' dt + \frac{[x]-x}{x}\cdot\frac{1}{\log x} + O(1)$$

$$= \log(\log x) + \int_2^{\infty} (t-[t])\left[\frac{1}{n\log n}\right]' dt + \int_{\infty}^{x} (t-[t])\left(\frac{1}{n\log n}\right)' dt + \frac{[x]-x}{x}\,\frac{1}{\log x} + O(1)$$

consider $\left|\frac{x-[x]}{x}\,\frac{1}{\log x}\right| \leq \frac{1}{x\log x}$

$$= O\left(\frac{1}{x\log x}\right)$$

consider $\left|\int_{\infty}^{x}(t-[t])\left[\frac{1}{n\log n}\right]' dt\right| \le \int_{\infty}^{x} d\left|\frac{1}{n\log n}\right| dt$

$$= \left[\frac{1}{n\log n}\right]_{\infty}^{x}$$

$$= \frac{1}{x\log x} = o\left(\frac{1}{x\log x}\right)$$

$$\therefore \sum_{2<n\le x} \frac{1}{n\log n} = \log(\log x) + B(\text{constant}) + o\left(\frac{1}{x\log x}\right)$$

where $B = \int_{2}^{\infty} \frac{x-[x]}{x} \frac{1}{\log x}$.

The Average order of the Division Function $\sigma(n)$

Theorem (1):- For $x\ge 1$, we have $\sum_{n\le x} \sigma_1(n) = \frac{1}{2}\xi(2)x^2 + o(x\log x)$ where $\xi(2) = \frac{\pi^2}{6}$, the average order is $\frac{n\pi^2}{12}$

Proof:- we know that $\sigma_1(n) = \sum_{q|n} q$

$$\sum_{n\le x} \sigma_1(n) = \sum_{n\le x} \sum_{q|n} q$$

since $q|n \Rightarrow n = q.d$

$$\sum_{n\le x} \sigma_1(n) = \sum_{d\le x} \sum_{q\le x/d} q$$

$$= \sum_{d\le x}\left[\frac{\left(\frac{x}{d}\right)^2}{1+1} + o\left(\frac{x}{d}\right)\right] \quad [\text{By condition(iv)}]$$

$$= \frac{x^2}{2}\sum_{d\le x}\frac{1}{d^2} + o\left(x\sum_{d\le x}\frac{1}{d}\right)$$

$$= \frac{x^2}{2}\left[\frac{x^{-1}}{-1} + \xi(2) + o(x^{-2})\right] + o\left[x\log x + c + o\left(\frac{1}{x}\right)\right]$$

$$= \frac{-x}{2} + \frac{x^2}{2}\xi(2) + o(1) + o(x\log x) + o(x) + o(1)$$

$$= o(x) + \frac{x^2}{2}\xi(2) + o(x\log x) \quad \left[\because \xi(2) = \pi^2/6\right]$$

The average order of $\sigma_1(n) = \frac{n\pi^2}{12}$

we know that $\bar{\sigma}_1(n) = \frac{1}{n}\sum_{n\le x}\sigma_1(n)$

$$= \frac{1}{n}\left[\frac{1}{2}\xi(2)n^2 + o(n\log n)\right]$$

$$= \frac{n}{2}\zeta(2) + O(\log n)$$

$$= \frac{n}{2}\frac{\pi^2}{6} + O(\log n)$$

$$\overline{\sigma_1}(n) = \frac{n\pi^2}{12} + O(\log n)$$

$$\Rightarrow \frac{\overline{\sigma_1}(n)}{\frac{n\pi^2}{12}} = 1 + \frac{12}{n\pi^2} O(\log n)$$

$$\lim_{n\to\infty} \frac{\overline{\sigma_1}(n)}{\frac{n\pi^2}{12}} = \lim_{n\to\infty}\left[1 + \frac{12}{n\pi^2} O(\log n)\right]$$

$$\lim_{n\to\infty} \overline{\sigma_1}(n) = \lim_{n\to\infty} \frac{n\pi^2}{12}$$

$$\therefore \overline{\sigma_1}(n) \approx \frac{n\pi^2}{12}$$

Theorem(2): For all $x \geq 1$, we have

$$\sum_{n\leq x} \sigma_\alpha(n) = \zeta\left[\frac{\alpha+1}{\alpha+1}\right]\frac{x^{\alpha+1}}{\alpha+1} + O(x^\beta) \text{ where } \beta = 1-\alpha \text{ and}$$

hence find the average order of $\sigma_\alpha(n)$.

Proof:- we know that $\sigma_\alpha(n) = \sum_{q|n} q^\alpha$

$$\sum_{n\leq x} \sigma_\alpha(n) = \sum_{n\leq x}\sum_{q|n} q^\alpha$$

$$= \sum_{d\leq x}\sum_{q\leq x/d} q^\alpha$$

$$= \sum_{d\leq x}\left[\frac{\left(\frac{x}{d}\right)^{\alpha+1}}{\alpha+1} + O\left(\frac{x}{d}\right)^\alpha\right] \quad [\text{By condition(iv)}]$$

$$= \frac{x^{\alpha+1}}{\alpha+1}\sum_{d\leq x}\frac{1}{d^{\alpha+1}} + O\left[x^\alpha \sum_{d\leq x}\frac{1}{d^\alpha}\right]$$

$$= \frac{x^{\alpha+1}}{\alpha+1}\left[\frac{x^{1-(\alpha+1)}}{1-(\alpha+1)} + \zeta(\alpha+1) + O\left[x^{-(\alpha+1)}\right]\right] +$$

$$O\left[x^\alpha \frac{x^{1-\alpha}}{1-\alpha} + \zeta(\alpha) + O(x^{-\alpha})\right) \quad [\text{By condition (ii)}]$$

$$= \frac{-x}{\alpha(\alpha+1)} + \frac{x^{\alpha+1}}{\alpha+1}\zeta(\alpha+1) + O\left(\frac{x}{1-\alpha}\right) + O(x^\alpha) + O\left[\zeta(\alpha)\right]$$

$$+ O(1)$$

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consider $\left|\dfrac{-x}{\alpha(\alpha+1)}\right| \leq |x| = O(x)$

$$\left|\frac{x}{1-\alpha}\right| \leq x = \frac{x}{1-\alpha}$$

$$= O(x)$$

$$\sum_{n\leq x} \sigma_\alpha(n) = O(x) + \frac{x^{\alpha+1}}{\alpha+1}\zeta(\alpha+1) + O(x) + O(x^\alpha) + O(1)$$

$$= \frac{x^{\alpha+1}}{\alpha+1}\zeta(\alpha+1) + O(x^\beta) \quad \text{where } \beta = \max\{1, \alpha\}$$

$\therefore$ The average order of σ_α is $\bar{\sigma}_\alpha(n) = \frac{1}{n}\sum_{n\leq x}\sigma_\alpha(n)$

$$\bar{\sigma}_\alpha(n) = \frac{1}{n}\left[\frac{n^{\alpha+1}}{\alpha+1}\zeta(\alpha+1) + O(n^\beta)\right]$$

$$= \frac{n^\alpha}{\alpha+1}\zeta(\alpha+1) + O(n^\beta)$$

$$\frac{\bar{\sigma}_\alpha(n)}{\frac{n^\alpha}{\alpha+1}\zeta(\alpha+1)} = 1 + \frac{\alpha+1}{n^\alpha}\frac{1}{\zeta(\alpha+1)}O(n^\beta)$$

$$\lim_{n\to\infty}\left[\frac{\bar{\sigma}_\alpha(n)}{\frac{n^\alpha}{\alpha+1}\zeta(\alpha+1)}\right] = \lim_{n\to\infty}\left[1 + \frac{\alpha+1}{n^\alpha\zeta(\alpha+1)}O(n^\beta)\right]$$

$$\Rightarrow \lim_{n\to\infty}\bar{\sigma}_\alpha(n) = \lim_{n\to\infty}\frac{n^\alpha}{\alpha+1}\zeta(\alpha+1)$$

$$\therefore \bar{\sigma}_\alpha(n) = \frac{n^\alpha}{\alpha+1}\zeta(\alpha+1)$$

Theorem ③: for all $\beta > 0$, let $\delta = \max\{0, 1, -\beta\}$ then

$$\sum_{n\leq x}\sigma_{-\beta}(n) = \zeta(\beta+1)x + O(x^\delta) \text{ if } \beta \neq 1$$

$$= \zeta(2)x + O(\log x) \text{ if } \beta = 1$$

Proof:- we know that $\sigma_\alpha(n) = \sum_{d/n} d^\alpha \quad [\because \alpha = -\beta]$

$$\sum_{n\leq x}\sigma_{-\beta}(n) = \sum_{n\leq x}\sum_{d/n} d^{-\beta}$$

$$= \sum_{d\leq x}\sum_{q\leq \frac{x}{d}}\frac{1}{d^\beta}$$

$$= \sum_{d\leq x}\frac{1}{d^\beta}\sum_{q\leq x/d} q^0$$

$$= \sum_{d \le x} \frac{1}{d^{\beta}}\left[\frac{x}{d} + O(1)\right]$$

$$= \sum_{d \le x} \frac{x}{d^{\beta+1}} + O\left[\sum_{d \le x} \frac{1}{d^{\beta}}\right] \quad \text{--- (1)}$$

$$= x\left[\frac{x^{1-\beta-1}}{1-\beta-1} + \zeta(\beta+1) + O(x^{-(\beta+1)})\right] + O\left[\frac{x^{1-\beta}}{1-\beta} + \zeta(\beta) + O(x^{-\beta})\right]$$

$$= \frac{x^{1-\beta}}{-\beta} + x\zeta(\beta+1) + O(x^{-\beta}) + O\left(\frac{x^{1-\beta}}{1-\beta}\right) + O[\zeta(\beta)] + O[x^{-\beta}]$$

consider $\left|\frac{x^{1-\beta}}{1-\beta}\right| \le |x^{1-\beta}|$

$$\Rightarrow \frac{x^{1-\beta}}{-\beta} = O(x^{1-\beta})$$

$$\sum_{n \le x} \sigma_{\beta}(n) = O(x^{1-\beta}) + \zeta(\beta+1) + O(x^{-\beta}) + O(x^{1-\beta}) + O(1)$$

$$= x\zeta(\beta+1) + O(x^{\delta}) \text{ where } \delta = \max\{0, 1, -\beta\}$$

From equation (1), we have $\sum_{n \le x} \sigma_{\beta}(x) = x\sum_{d \le x} \frac{1}{d^{\beta+1}} + O\left[\sum_{d \le x} \frac{1}{d^{\beta}}\right]$

$$= x\sum_{d \le x} \frac{1}{d^2} + O\left[\sum_{d \le x} \frac{1}{d}\right]$$

$$= x\left[\frac{x^{1-2}}{1-2} + \zeta(2) + O(x^{-2})\right] + O\left[\log x + c + O\left(\frac{1}{x}\right)\right]$$

$$= -1 + x\zeta(2) + O\left(\frac{1}{x}\right) + O(\log x) + O\left(\frac{1}{x}\right)$$

$$= O(1) + x\zeta(2) + O\left(\frac{1}{x}\right) + O(\log x)$$

$$= x\zeta(2) + O(\log x)$$

$$\therefore \sum_{n \le x} \sigma_{\beta}(n) = \zeta(\beta+1)x + O(x^{\delta}) \text{ if } \beta \ne 1$$

$$= x\zeta(2) + O(\log x) \text{ if } \beta = 1$$

Hence the proof.

Note: The average order of $\phi(n)$

The asympotic formula for the partial sums of euler φ totient function involves in the sum of the series $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$

we know that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \sum_{n\le x} \frac{\mu(n)}{n^2} + \sum_{n> x} \frac{\mu(n)}{n^2}$$

$$\sum_{n\le x} \frac{\mu(n)}{n^2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} - \sum_{n> x} \frac{\mu(n)}{n^2} \quad [\mu(n) = \pm 1]$$

$$= \frac{6}{\pi^2} - \sum_{n> x} \frac{1}{n^2}$$

$$= \frac{6}{\pi^2} - O(x^{1-2})$$

$$= \frac{6}{\pi^2} - O\left(\frac{1}{x}\right)$$

Theorem ④:- for $x \ge 1$, we have $\sum_{n\le x} \phi(n) = \frac{3x^2}{\pi^2} + O(x\log x)$ and the average order is $\frac{3n}{\pi^2}$

Proof:-

From the relation connecting ϕ and μ

$$\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$$

$$\sum_{n\le x} \phi(n) = \sum_{n\le x} \sum_{d|n} \mu(d)\frac{n}{d}$$

$$\therefore d|n \Rightarrow n = dq$$

$$= \sum_{d\le x} \sum_{q\le \frac{x}{d}} \mu(d)q$$

$$= \sum_{d\le x} \mu(d) \sum_{q\le \frac{x}{d}} q \quad [\because \text{By condition iv}]$$

$$= \sum_{d\le x} \mu(d)\left[\left(\frac{x}{d}\right)^2 \frac{1}{2} + O\left(\frac{x}{d}\right)\right]$$

$$= \frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2} + O\left[\sum_{d\le x}\mu(d)\frac{x}{d}\right]$$

$$\le \frac{x^2}{2}\left[\frac{6}{\pi^2} - O\left(\frac{1}{x}\right)\right] + O\left[x\sum_{d\le x}\frac{1}{d}\right] \quad [\because \mu(d)=1]$$

$$= \frac{3x^2}{\pi^2} - O\left(\frac{x}{2}\right) + O\left[x\left(\log x + c + O\left(\frac{1}{x}\right)\right)\right] \quad [\text{By}(i)]$$

$$= \frac{3x^2}{\pi^2} + O(x\log x) + O(x) + O(1)$$

$$= \frac{3x^2}{\pi^2} + O(x\log x)$$

Average order $\underline{\overline{\phi}(n)} = \frac{1}{n}\sum_{n\le x}\phi(n)$

$$\overline{\phi}(n) = \frac{1}{n}\left[\frac{3n^2}{\pi^2} + O(n\log n)\right]$$

$$= \frac{3n}{\pi^2} + O(\log n)$$

$$\lim_{n\to\infty}\frac{\overline{\phi}(n)}{\frac{3n}{\pi^2}} = \lim_{n\to\infty}\left[1 + \frac{O(\log n)}{\frac{3n}{\pi^2}}\right]$$

$$\lim_{n\to\infty}\overline{\phi}(n) = \lim_{n\to\infty}\frac{3n}{\pi^2}$$

$$\overline{\phi}(n) = \frac{3n}{\pi^2}$$

Sec: An Application of the Distribution of Lattice points visible from the origin

Definition:- Two lattice points p and q are said to be mutually visible. If the line segment joining p and q doesnot contains any lattice points other than the end points.

Theorem ⑤:- Two lattice points (a,b) and (m,n) are mutually visible iff $(a-m, b-n)$ are relatively prime

Proof:- We have to P.T (a,b) and (m,n) are mutually visible $\Leftrightarrow (a-m, b-n) = 1$

Now we prove the theorem for $(m,n) = (0,0)$

i.e., To S.T (a,b) and $(0,0)$ are mutually visible $\Leftrightarrow$

$(a,b) = 1$

Suppose (a,b) and $(0,0)$ are mutually visible

Claim:- $GCD(a,b) = 1$

If possible, suppose that $(a,b) = d$

$\Rightarrow \exists\, a'$ and b' such that $a = a'd$ and $b = b'd$ where

$(a', b') = 1$

i.e., the point (a', b') lies on the line segment joining

(a,b) and $(0,0)$

$\Rightarrow (a,b)$ and $(0,0)$ are not mutually visible

which is a contradiction

$(a,b) = 1$

Conversely suppose that $(a,b) = 1$

Now we have to S.T (a,b) & $(0,0)$ are mutually visible

let (a', b') be a lattice points which lies on the line segment joining (a,b) and $(0,0)$

$\Rightarrow a' = ta, b' = tb$ where $0 < t < 1$

$\therefore t$ is a rational number

$\therefore t = \frac{r}{s}$ where $(r,s) = 1$

$\therefore a' = \frac{r}{s} a$ and $b' = \frac{r}{s} b$

$\Rightarrow sa' = ra$ and $sb' = rb$

$\Rightarrow s | ra$ and $s | rb$

$\Rightarrow s | a$ and $s | b$

$\Rightarrow (a,b) = s$

But $(a,b) = 1 \Rightarrow s = 1$

$\therefore t = \frac{r}{s} \Rightarrow t = r$

$\therefore$ 't' is not a rational number which is a contradiction

$\therefore$ The points (a,b) and $(0,0)$ are mutually visible.

Partial sums of Dirichlet Product

Theorem (6):- let 'f' and 'g' be arithmetic functions $h = f * g$ if $H(x) = \sum_{n \le x} h(n)$; $G(x) = \sum_{n \le x} g(n)$ and $F(n) = \sum_{n \le x} f(n)$ then $H(x) = \sum_{n \le x} f(n)\, G\left(\frac{x}{n}\right) = \sum_{n \le x} g(n) F\left(\frac{x}{n}\right)$.

Note:-

In order to prove this theorem, we have to use associative law relatively 'b' and $*$

i.e., $\alpha \circ (\beta \circ f) = (\alpha * \beta) \circ f$

we know that $\mu(n) = \begin{cases} 1 & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$

Proof:- Given that $H(x) = \sum_{n \le x} h(n) \cdot 1$

$$= \sum_{n \le x} h(n) \cdot \mu\left(\frac{x}{n}\right)$$

$$= h \circ \mu(x)$$

$$H = h \circ \mu$$

$$G(x) = \sum_{n \le x} g(n) \mu\left(\frac{x}{n}\right)$$

$$= g \circ \mu(x)$$

$$= g \circ \mu$$

$$G = g \circ \mu$$

Given that $F(x) = \sum_{n \le x} f(n) \cdot 1$

$$= \sum_{n \le x} f(n) \mu\left(\frac{x}{n}\right)$$

$$= f \circ \mu(x)$$

$$F = f \circ \mu$$

consider $f \circ G = f \circ (g \circ \mu)$

$$= (f * g) \circ \mu$$

$$= h o \mu$$

$$f o G(x) = h o \mu(x)$$

$$= \sum_{n \le x} h(n) \mu\left(\frac{x}{n}\right) \qquad \left[\therefore \mu\left(\frac{x}{n}\right) = 1 \right]$$

$$= \sum_{n \le x} h(n)$$

$$f o G(x) = H(x)$$

$$H(x) = f o G(x) = \sum_{n \le x} f(n) G\left(\frac{x}{n}\right)$$

consider $g o F = g o (f o \mu)$

$$= (g * f) o \mu$$

$$= h o \mu$$

$$g o F(x) = h o \mu(x)$$

$$= \sum_{n \le x} h(n) \mu\left(\frac{x}{n}\right)$$

$$= \sum_{n \le x} h(n)$$

$$= H(x)$$

$$H(x) = g o F(x)$$

$$= \sum_{n \le x} g(n) F\left(\frac{x}{n}\right)$$

$$H(x) = \sum_{n \le x} f(n) G\left(\frac{x}{n}\right) = \sum_{n \le x} g(n) F\left(\frac{x}{n}\right)$$

Hence proved.

Theorem (7): If $F(x) = \sum_{n \le x} f(n)$ then $\sum_{n \le x} \sum_{d \mid n} f(d) = \sum_{n \le x} f(n) \left[\frac{x}{n}\right] = \sum_{n \le x} F\left(\frac{x}{n}\right)$.

Proof:

Recall: By above theorem

$$H(x) = \sum_{n \le x} f(n) G\left(\frac{x}{n}\right) = \sum_{n \le x} g(n) F\left(\frac{x}{n}\right) \longrightarrow (1)$$

Take $g(n)=1$ then

$$G(x)=\sum_{n\le x} g(n)$$

$$=\sum_{n\le x} 1=[x] \longrightarrow ②$$

$$H(x)=\sum_{n\le x} f(n)G\left(\frac{x}{n}\right)=f(n)\left(\frac{x}{n}\right)=\sum_{n\le x} F\left(\frac{x}{n}\right) \longrightarrow ③$$

Now it is enough to p.T $\sum_{n\le x}\sum_{d|n}[f(d)]=\sum_{d\le x}\sum_{q\le \frac{x}{d}} f(d)$

$$=\sum_{d\le x} f(d)\sum_{q=\frac{x}{d}}(1)$$

$$=\sum_{d\le x} f(d)\left[\frac{x}{d}\right]$$

$$=\sum_{n\le x} f(n)\left[\frac{x}{n}\right]=H(x)$$

$$\therefore ③ \Rightarrow \sum_{n\le x}\sum_{d|n} f(d)=\sum_{n\le x} f(n)\left[\frac{x}{n}\right]=\sum_{n\le x} F\left[\frac{x}{n}\right]=H(x)$$

Hence proved.

Theorem :- For $x\ge 1$, we have $\sum_{n\le x}\mu(n)\left[\frac{x}{n}\right]=1$ and

$\sum_{n\le x}\Lambda(n)\left[\frac{x}{n}\right]=\log[x]!$

Proof :-

Recall :- By above theorem

$$\sum_{n\le x}\sum_{d|n} f(d)=\sum_{n\le x} f(n)\left[\frac{x}{n}\right]=\sum_{n\le x} F\left(\frac{x}{n}\right) \longrightarrow ①$$

Take $f(n)=\mu(n)$ in eq^n ①

$$\sum_{n\le x}\sum_{d|n}\mu(d)=\sum_{n\le x}\mu(n)\left[\frac{x}{n}\right] \qquad \left[\because \sum_{d|n}\mu(d)=\frac{1}{n}\right]$$

$$\sum_{n\le x}\left[\frac{1}{n}\right]=\sum_{n\le x}\mu(n)\left[\frac{x}{n}\right]$$

$$\therefore \sum_{n\le x}\mu(n)\left[\frac{x}{n}\right]=1$$

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Take $f(n) = \Lambda(n)$ in equation ①

$$\therefore \sum_{n \le x} \sum_{d|n} \Lambda(d) = \sum_{n \le x} \Lambda(n) \left[\frac{x}{n}\right]$$

$$\sum_{n \le x} \Lambda(n) \left[\frac{x}{n}\right] = \sum_{n \le x} \log n = \sum_{n=1}^{x} \log n$$

$$= \log 1 + \log 2 + \dots + \log x$$

$$= \log(1 \cdot 2 \cdot \dots \cdot x)$$

$$= \log [x]!$$

$$\therefore \sum_{n \le x} \Lambda(n) \left[\frac{x}{n}\right] = \log [x]!$$

Hence the proof

Note:- $2 \le n \le x \left\{\frac{x}{n}\right\} = [x] - 1$

Theorem:- $\forall x \ge 1$ we have $\left|\sum_{n \le x} \frac{\mu(n)}{(n)}\right| \le 1$ if $x \ge 2$

and $\left|\sum_{n \le x} \frac{\mu(n)}{n}\right| = 1$ if $x < 2$.

Proof:-

if $x < 2$ then there is only one term in the summation

i.e., $x = 1$

$$\therefore \left|\sum_{n \le x} \frac{\mu(n)}{n}\right| = \left|\sum_{n=1} \frac{\mu(n)}{n}\right| = \left|\frac{\mu(1)}{1}\right| = 1$$

we have $(x) = [x] + \{x\}$ where $0 < \{x\} < 1$

if $x \ge 2$, from the above theorem, we have

$$\Rightarrow \sum_{n \le x} \mu(n) \left[\frac{x}{n}\right] = 1$$

$$\Rightarrow \sum_{n \le x} \mu(n) \left[\left(\frac{x}{n}\right) - \left\{\frac{x}{n}\right\}\right] = 1$$

$$\Rightarrow \sum_{n \le x} \mu(n) \left(\frac{x}{n}\right) - \sum_{n \le x} \mu(n) \left\{\frac{x}{n}\right\} = 1$$

$$\left|\sum_{n \le x} \mu(n) \left(\frac{x}{n}\right)\right| \le \left|1 + \sum_{n \le x} \mu(n) \left\{\frac{x}{n}\right\}\right|$$

$$\le \left|1 + \mu(1) \left\{\frac{x}{n}\right\}\right|$$

$$\le \left|1 + \mu(1) \{x\} + \sum_{2 \le n \le x} \mu(n) \frac{\{x\}}{n}\right|$$

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$$\leq \left| 1 + \{x\} + [x] - 1 \right|$$

$$\leq |x|$$

$$\therefore \left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq 1$$

Legenders Identity:- For $x \geq 1$, we have $[x]! = \prod_{p \leq x} p^{\alpha(p)}$ where the product is extended over all the primes $\alpha \leq x$ and $\alpha(p) = \sum_{m=1}^{\infty} \left[\frac{x}{p^m}\right] \rightarrow (1)$

where $\alpha(p)$ is finite. since $\left[\frac{x}{p^m}\right] = 0$ for $p > x$

Proof:- By known theorem, we have - Th (6) recall if essay

$$\log[x]! = \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] \rightarrow (2)$$

where $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, m > 1 \\ 0 & \text{otherwise} \end{cases}$

$n = p^m$ in eq^n (2)

$$\log[x]! = \sum_{p^m \leq x} \Lambda(p^m) \left[\frac{x}{p^m}\right]$$

$$= \sum_{p \leq x} \sum_{m=1}^{\infty} \log p \left[\frac{x}{p^m}\right]$$

$$= \sum_{m=1}^{\infty} \left[\frac{x}{p^m}\right] \sum_{p \leq x} \log p$$

$$= \sum_{p \leq x} \log p \cdot \sum_{m=1}^{\infty} \log\left(\frac{x}{p^m}\right)$$

$$= \sum_{p \leq x} \log p \; \alpha(p) \quad [\because \text{from eq}^n (1)]$$

$$= \sum_{p \leq x} \alpha(p) \log p = \alpha(p) \sum_{p \leq x} \log p$$

$$= \sum_{p \leq x} \log p^{\alpha(p)}$$

$$\log[x]! = \log \left[\prod_{p \leq x} p^{\alpha(p)}\right]$$

$$[x]! = \prod_{p \leq x} p^{\alpha(p)}$$

Hence the proof.

Theorem:- If $x \geq 2$, we have $\log[x]! = x\log x - x + O(\log x)$.

and hence $\sum_{n \leq x} \Lambda(n)\left[\frac{x}{n}\right] = x\log x - x + O(\log x)$.

Proof:- consider $\log[x]! = \log[1 \cdot 2 \cdot 3 \cdots [x]]$

$= \log 1 + \log 2 + \cdots + \log x$

$= \sum_{n \leq x} \log n$ —— ①

By Euler's summation formula, we have

$$\sum_{y < n \leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t - [t]) f'(t)\,dt + ([x] - x) f(x) - ([y] - y) f(y)$$

Take $f(x) = \log x$, $f'(x) = \frac{1}{x}$

$$\sum_{n \leq x} \log n = \int_y^x \log t\,dt + \int^x (t - [t])\frac{1}{t}\,dt + ([x] - x)\log x + O(1)$$

$$= x\log x - x + O(1) + \int_1^x (t - [t])\frac{1}{t}\,dt + ([x] - x)\log x + O(1)$$

consider $|([x] - x)\log x| \leq |\log x|$

$\therefore ([x] - x)\log x = O(\log x)$

consider $\left|\int_1^x (t - [t])\frac{1}{t}\,dt\right| \leq \left|\int_1^x \frac{1}{t}\,dt\right|$

$\leq |\log t|_1^x = \log x - \log 1$

$\therefore \int^x (t - [t])\frac{1}{t}\,dt = O(\log x)$

$\therefore \sum_{n \leq x} \log n = x\log x - x + O(1) + O(\log x) + O(\log x) + O(1)$

$\log[x]! = x\log x - x + O(\log x)$ [$\because$ By ①]

By known theorem, we have $\log[x]! = \sum_{n \leq x} \Lambda(n)\left[\frac{x}{n}\right]$

$$\sum_{n \leq x} \Lambda(n)\left[\frac{x}{n}\right] = x\log x - x + O(\log x)$$

Theorem:- If $x \geq 2$, we have $\sum_{p \leq x}\left[\frac{x}{p}\right]\log p = x\log x + O(x)$

where the sum is extended over all the primes $p \leq x$

Proof:-

consider $\sum_{n \leq x}\left(\frac{x}{n}\right)\Lambda(n)$

Take $n=p^m$

$$\Rightarrow \sum_{p^m\le x}\left[\frac{x}{p^m}\right]\Lambda(p^m)$$

$$\Rightarrow \sum_{p\le x}\sum_{m=1}^{\infty}\left[\frac{x}{p^m}\right]\Lambda(p^m)$$

$$\Rightarrow \sum_{p\le x}\left(\frac{x}{p}\right)\log p+\sum_{p\le x}\sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right]\log p \longrightarrow ①$$

consider $\sum_{p\le x}\sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right]\log p = \sum_{p\le x}\log p \cdot \sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right]$

$$= x\sum_{p\le x}\log p\sum_{m=2}^{\infty}\frac{1}{p^m}$$

$$= x\sum_{p\le x}\log p\left[\frac{1}{p^2}+\frac{1}{p^3}+\cdots\right]$$

$$= x\sum_{p\le x}\log p\,\frac{1}{p^2}\left[1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right]$$

$$= x\sum_{p\le x}\log p\,\frac{1}{p^2}\left[\frac{1}{1-\frac{1}{p}}\right]$$

$$= x\sum_{p\le x}\frac{\log p}{p(p-1)} \quad \left[\because \left|x\sum_{n\le x}\frac{\log n}{x^2 n(n-1)}\right| \le |x| = O(x)\right]$$

$$= x\sum_{n\le x}\frac{\log n}{n(n-1)}$$

$$\sum_{p\le x}\sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right]\Lambda(p^m) = O(x) \longrightarrow ②$$

substitute ② in ①

$$\sum_{n\le x}\left[\frac{x}{n}\right]\Lambda(n) = \sum_{p\le x}\left[\frac{x}{p}\right]\log p + O(x)$$

$$\Rightarrow \sum_{p\le x}\left[\frac{x}{p}\right]\log p = \sum_{n\le x}\left[\frac{x}{n}\right]\Lambda(n) + O(x)$$

$$= x\log x - x + O(\log x) + O(x)$$

$$= x\log x + O(x) + O(\log x)$$

$$\therefore \sum_{p\le x}\left[\frac{x}{p}\right]\log p = x\log x + O(x)$$

Another identity for the partial sums of Dirchilet product

We write $F(x) = \sum_{n \le x} f(n)$

$$G(x) = \sum_{n \le x} g(n)$$

$$H(x) = \sum_{n \le x} (f * g)(n)$$

so that $H(x) = \sum_{n \le x} \sum_{d | n} f(d) g\left(\frac{n}{d}\right)$

$$= \sum_{\substack{q|d \\ qd \le x}} f(d) g(q)$$

Theorem:- If 'a' and 'b' are positive real numbers such that $ab = 1$ then $\sum_{\substack{q|d \\ qd \le x}} f(d) g(q) = \sum_{n \le a} f(n) g\left(\frac{x}{n}\right) + \sum_{n \le b} g(n) F\left(\frac{x}{n}\right) - F(a) G(b)$

Proof:- Given that 'a' and 'b' are positive real numbers $\ni ab = 1$

$$F(x) = \sum_{n \le x} f(n) \longrightarrow ①$$

$$G(x) = \sum_{n \le x} g(n) \longrightarrow ②$$

$$H(x) = \sum_{n \le x} (f * g)(n)$$

so that $H(x) = \sum_{n \le x} \sum_{d | n} f(d) g\left(\frac{n}{d}\right)$

$$= \sum_{q|d} f(d) g(q) \longrightarrow ③$$

This sum $H(x)$ extended over the lattice points in the hyperbolic region

We split the sum into ~~two points~~ parts, one over the points $A \cup B$ and other over those in $B \cup C$

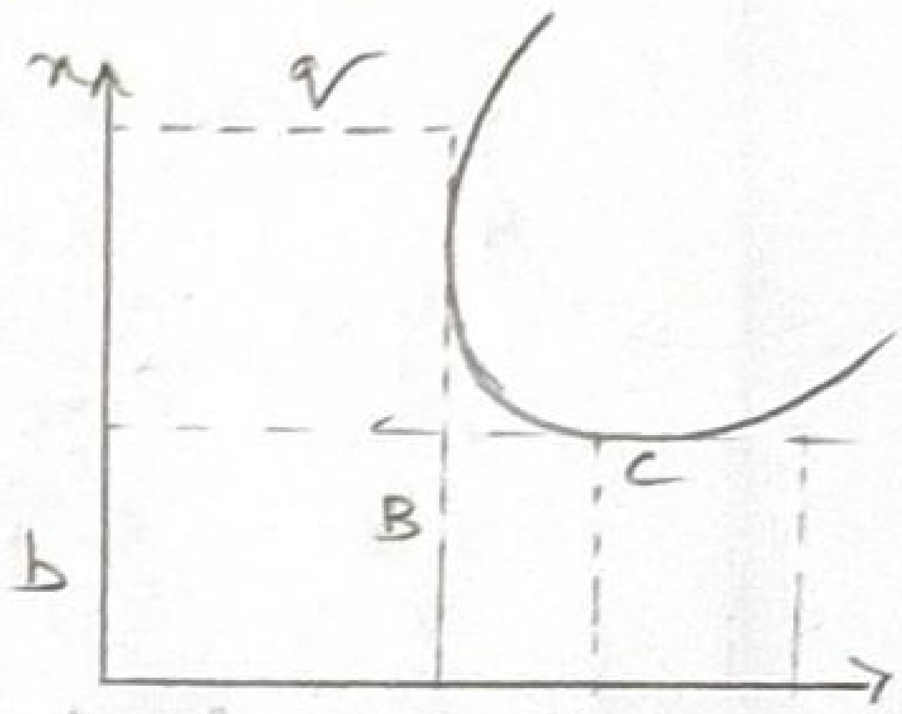
The lattice points in B are covered twice so we have,

$$H(x) = \sum_{d \le a} \sum_{q \le x/d} f(d)g(q) + \sum_{q \le b} \sum_{d \le x/q} f(d)g(q) - \sum_{d \le a} \sum_{q \le b} f(d)g(q)$$

$$= \sum_{d \le a} f(d) \sum_{q \le x/d} g(q) + \sum_{d \le x/q} f(d) \sum_{q \le b} g(q) - \sum_{d \le a} f(d) \sum_{q \le b} g(q)$$

$$= \sum_{d \le x} f(d) g\left(\frac{x}{d}\right) + \sum_{q \le b} g(q) f\left(\frac{x}{q}\right) - F(a)G(b)$$

$$= \sum_{n \le a} f(n) g\left(\frac{x}{n}\right) + \sum_{n \le b} g(n) F\left(\frac{x}{n}\right) - F(a)G(b)$$



Theorem:- The set of lattice points visible from the origin has identity $6/\pi^2$

Proof:- Consider a square region in xy-plane $|x| \le r$, $|y| \le r$

let $N(r)$ denotes the number of lattice points in the square and let $N'(r)$ denotes the number of lattice points visible from the origin

$\lim_{n \to \infty} \frac{N'(r)}{N(r)}$ gives the density of lattice points visible from the origin

claim:- $\lim_{n \to \infty} \frac{N'(r)}{N(r)} = \frac{6}{\pi^2}$

There are 8 lattice points near the origin and visible from the origin

By the symmetry, number of lattice points visible from the origin in 8+8 times the number of lattice points in the region
$[(x, y),\ 2 \le x \le r,\ 1 \le y \le x]$

i.e., shaded region.

$N'(r) = 8+8$ times the number of lattice points in the region $\{(x,y),\ 2 \le x \le r,\ 1 \le y \le x\} = 8+8\left\{\sum_{2\le n\le r} \sum_{\substack{1\le m< r\\(m,n)=1}}\right\}$

$$= 8\phi(1)+8\sum_{2\le n\le r}\left[\sum_{\substack{1\le m\le n\\(m,n)=1}}\right]$$

$$= 8\phi(1)+8\sum_{2\le n\le r}\phi(n)$$

$$= 8\sum_{1\le n\le r}\phi(n) \quad \text{[By previous theorem]}$$

$$= 8\left[\frac{3r^2}{\pi^2}+O(r\log r)\right]$$

$$= \frac{24r^2}{\pi^2}+O(r\log r)$$

Now we find $N(r)$ taking the number of lattice in the square $|x|\le r$, $|y|\le r$

We have 'r' lattice points on x-axis and 'r' lattice points on y-axis and the origin

$\therefore$ The number of lattice points on the side $[2(r)+1]$

$$\therefore N(r) = [2(r)+1]^2 = [2r+O(1)]^2$$

$$= 4r^2+O(1)+4rO(1)$$

$$= 4r^2+O(r)$$

$$\lim_{r\to\infty}\frac{N'(r)}{r} = \lim_{r\to\infty}\left[\frac{\frac{24r^2}{\pi^2}+O(r\log r)}{4r^2+O(r)}\right]$$

$$= \lim_{r\to\infty}\frac{r^2\left[\frac{24}{\pi^2}+O\left(\frac{\log r}{r}\right)\right]}{r^2\left[4+O\left(\frac{1}{r}\right)\right]}$$

$$= \frac{24}{\pi^2} \times \frac{1}{4} = \frac{6}{\pi^2}$$

Hence the proof.